Stochastic Differential Equations for Power Law Behaviors

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***Abstract*— In this paper we present simple stochastic differential equations that lead to lower-tail and/or upper tail power law behaviors. We also present a model with bi-directional Poisson counters that exhibits power law behavior near a critical point, which might be of interest to statistical physics.**

I. INTRODUCTION

It has been observed many times in recent years that empirical studies dealing with a variety of subject matter produce data showing power law histograms extending over several decades or more. The diversity of application areas, and the extent of the data have inspired researchers to look for general principles that would make it possible to trace the individual phenomena back to one or more common features. Among recent work along these lines we call attention to [1], [4], [5], [6]. The behavior of a distribution $F(x)$ as $x \to \infty$ is referred to as the upper tail behavior. When the distribution has its support lower bounded by B, its behavior as $x \downarrow B$ is referred to as the lower tail behavior. In this paper we pursue the idea that certain simple forms of first order stochastic differential equations have steady state densities which show lower tail and/or upper tail power law behaviors, depending on the values of the parameters. To the extent that the form and parameters of a differential equation are often more readily identified in the modeling process, the differential equations we describe may be thought of as providing a more direct explanation of power law behavior.

In their interesting paper [6], Reed and Hughes consider the probability density of the value of an exponentially growing quantity sampled at an exponentially distributed random time. They observe that this generates a power law distribution. Here we consider a different situation involving the steady state density associated with a stochastic differential equation. The equation describes a situation in which the quantity of interest decays to zero exponentially, $\dot{X} = -\alpha X$, but is incremented by a fixed amount σ at random times, the times having an exponential distribution. We show that for a range of parameter values the steady state distribution of X exhibits a power law lower tail close to zero, which is the lower end of its range. The fact that we deal with the steady state property of an ongoing dynamics gives our work a different set of possibilities for interpretation. The basic reason as to why the distribution of values in steady state has the same form as the distribution of values obtained by sampling at a random time, lies in the form of the drift term associated with the differential equation.

The lower tail behavior is of interest when we deal with a quantity that is intrinsically bounded from below, such as queue length. Our models for lower tail behavior focuses on the steady state behavior associated with stochastic differential equations (SDEs) containing a Poisson counter N and taking the form Our models for the lower tail behavior focuses on the steady state behavior associated with stochastic differential equations (SDEs) containing a Poisson counter N and taking the form

$$\mathrm{d}X = f(X)\mathrm{d}t + g(X)\mathrm{d}N.$$

A different but related situation is that the range of the quantity of interest extends in both directions from a critical value, as might be the situation for some types of populations near a phase transition. For behaviors near a critical point, we consider slightly more complicated models of the form

$$\mathrm{d}X = f(X)\mathrm{d}t + g_1(X)\mathrm{d}N_1 - g_2(X)\mathrm{d}N_2.$$

We also study the more popular case of upper tail power law as the value of the quantity of interest goes to infinity. A simple transformation is shown to convert lower tail power laws into upper tail power laws, which also correspond to steady state densities of SDEs. In our view, these SDEs provide suitable generative interpretations of many power law upper tails observed in real data.

In Section II we first present our motivating example of a simple Poisson counter driven SDE (PCSDE), the steady-state distribution of which was shown by Brockett [1] to exhibit power law behavior near the origin. We then introduce a simple transformation of this PCSDE to develop an SDE that leads to an upper tail power law. We observe that a variety of distributions can be generated via simple modifications of the "drift term" in the SDE. In Section III, we add a Brownian motion term to obtain a similar result as Reed [5]. All of these cases demonstrate that random multiplication with exponential stopping time leads to power law behaviors. In Section IV we develop an SDE driven by Poisson counters in both positive and negative directions. We show that the steady-state distribution can exhibit power law behavior near a critical point. This may have implications in statistical physics since a discontinuity occurs in a surprising way. Section V concludes the paper.

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II. SDE DRIVEN BY A POISSON COUNTER

The following SDE was considered by Brockett [1],

$$\mathrm{d}X_t = -\alpha X_t \mathrm{d}t + \sigma \mathrm{d}N_t, \tag{1}$$

where $\alpha, \sigma > 0$ and N is a Poisson process of intensity λ. By Theorem 6.2 in [2], there is a unique adapted RCLL (right-continuous with left limits, or, càdlàg) process $\{X_t\}$ satisfying (1) and

$$\sup_{t\in[0,T]} \mathbb{E}[X_t^2] < \infty \tag{2}$$

for any $T \in [0, \infty)$. Similar arguments for the existence of solutions apply to all the other SDE's considered in this paper and will not be repeated.

Now let $\psi_k(x) = e^{ikx}, k \in \mathbb{R}$. By Itô's formula,

$$\begin{aligned}\mathrm{d}\psi_k(X_t) =& - i\alpha k X_t \psi_k(X_t)\mathrm{d}t \\ &+ [\psi_k(X_{t-} + \sigma) - \psi_k(X_{t-})]\mathrm{d}N_t \\ =& - \alpha k \frac{\partial \psi_k(X_t)}{\partial k}\mathrm{d}t + (e^{ik\sigma} - 1)\psi_k(X_{t-})\mathrm{d}N_t.\end{aligned}$$

Taking expectations,

$$\frac{\partial}{\partial t}\Phi_X(k,t) = -\alpha k \frac{\partial}{\partial k}\Phi_X(k,t) + \lambda(e^{ik\sigma} - 1)\Phi_X(k,t), \tag{3}$$

where $\Phi_X(k,t) = \mathbb{E}[\psi_k(X_t)]$ is the characteristic function of X_t and the change of the order of differentiation and expectation can be justified by (2) and Lebesgue's Dominated Convergence Theorem. Equation (3) can be solved by the method of characteristics (see e.g. [3]) to yield

$$\Phi_X(k,t) = \Phi_X(ke^{-\alpha t}, 0)\exp\left\{\lambda\int_0^t \left[e^{ik\sigma e^{\alpha(s-t)}} - 1\right]\mathrm{d}s\right\}. \tag{4}$$

After a change of variable $u = \sigma\exp(\alpha(s-t))$, (4) becomes

$$\Phi_X(k,t) = \Phi_X(ke^{-\alpha t}, 0)\exp\left\{\frac{\lambda}{\alpha}\int_{\sigma e^{-\alpha t}}^{\sigma} \frac{e^{iku} - 1}{u}\mathrm{d}u\right\}, \tag{5}$$

yielding

$$\Phi_X(k,\infty) = \exp\left\{\frac{\lambda}{\alpha}\int_0^{\sigma} \frac{e^{iku} - 1}{u}\mathrm{d}u\right\}.$$

This result can also be obtained as a consequence of Theorem 2 of [10]. By Lemma 53.2 of [7], the steady-state distribution is absolutely continuous with continuous density on $(0, \infty)$. Moreover, the density is continuously differentiable on $(0, \infty)$ if $\lambda > \alpha$.

Note that $\Phi_X(k,\infty)$ satisfies the following equation, obtained from (3) by setting the right-hand side to zero,

$$-\alpha k \frac{\mathrm{d}}{\mathrm{d}k}\Phi(k) + \lambda(e^{ik\sigma} - 1)\Phi(k) = 0.$$

Thus the corresponding density $f_X(x)$ satisfies

$$\alpha\frac{\mathrm{d}}{\mathrm{d}x}[xf(x)] + \lambda f(x-\sigma) - \lambda f(x) = 0. \tag{6}$$

It was shown in [9] that $f_X(x) = 0$ for $x < 0$. We can arrive at the same conclusion in a more intuitive way by examining (1). Indeed, note that, in spite of its initial value, X will eventually become positive and remain so from that point on. Now using $f_X(x) = 0$ for $x \le 0$, (6) can be solved recursively to give

$$f_X(x) = Cx^{\frac{\lambda}{\alpha}-1}, \quad x \in (0, \sigma],$$

and, for $x \in (n\sigma, n\sigma + \sigma]$, $n \ge 1$,

$$f_X(x) = f_X(n\sigma)\left(\frac{x}{n\sigma}\right)^{\frac{\lambda}{\alpha}-1} - \frac{\lambda}{\alpha}x^{\frac{\lambda}{\alpha}-1}\int_{n\sigma}^{x} u^{-\frac{\lambda}{\alpha}} f_X(u-\sigma)\mathrm{d}u,$$

where the constant C is determined by the normalization condition

$$\int_{-\infty}^{\infty} f_X(x)\mathrm{d}x = 1.$$

Note that $f_X(x)$ has a power law at its lower tail.

We briefly mention that (1) has the following generalization,

$$\mathrm{d}\mathbf{X}_t = \mathbf{A}\mathbf{X}_t \mathrm{d}t + \mathbf{b}\mathrm{d}N_t,$$

where $\mathbf{X}$ is an $\mathbb{R}^n$-valued process, $\mathbf{A}$ an $n \times n$ stable matrix and $\mathbf{b} \in \mathbb{R}^n$. Then the steady-state distribution has the following characteristic function

$$\Phi_{\mathbf{X}}(\mathbf{k},\infty) = \exp\left\{\lambda\int_0^{\infty}\left[\exp\left(i\mathbf{k}^T e^{\mathbf{A}s}\mathbf{b}\right) - 1\right]\mathrm{d}s\right\},$$

where $\mathbf{k} \in \mathbb{R}^n$, and the projection of $\mathbf{X}$ onto a left eigenvector of $\mathbf{A}$ exhibits power law near the origin.

Now we introduce the simple transformation $Y_t = X_t^{-1}$ to convert the lower tail power law into an upper tail power law. For $y \ge \varepsilon \triangleq \sigma^{-1}$, the steady-state density of Y is

$$f_Y(y) = f_X(y^{-1})y^{-2} = Cy^{-\frac{\lambda}{\alpha}-1}, \quad y \in [\varepsilon, \infty).$$

Using Itô's formula, we obtain the equation governing the evolution of Y_t,

$$\mathrm{d}Y_t = \alpha Y_t \mathrm{d}t - \frac{Y_{t-}^2}{\varepsilon + Y_{t-}}\mathrm{d}N_t, \tag{7}$$

Note that at each jumping point, Y drops to $\frac{\varepsilon Y_{t-}}{\varepsilon + Y_{t-}}$, which depends on Y_{t-}.

We can modify the coefficient in front of $\mathrm{d}N_t$ in (7) so that it always restores the process to a fixed point. The equation then becomes

$$\mathrm{d}Z_t = \alpha Z_t \mathrm{d}t + (z_0 - Z_{t-})\mathrm{d}N_t, \tag{8}$$

with the corresponding equation for the characteristic function $\Phi_Z(k,t)$ of Z_t being

$$\frac{\partial}{\partial t}\Phi_Z(k,t) = \alpha k\frac{\partial}{\partial k}\Phi_Z(k,t) - \lambda\Phi_Z(k,t) + \lambda e^{ikz_0}. \tag{9}$$

Equation (9) can be solved again by the method of characteristics, yielding

$$\begin{aligned}\Phi_Z(k,t) =& e^{-\lambda t}\Phi_Z(ke^{\alpha t}, 0) \\ &+ \lambda\int_0^t \exp\left\{\lambda(s-t) + iz_0 k e^{\alpha(t-s)}\right\}ds.\end{aligned} \tag{10}$$

After a change of variable $z = z_0 e^{\alpha(t-s)}$, (10) becomes

$$\Phi_Z(k,t) = e^{-\lambda t}\Phi_Z(ke^{\alpha t},0) + \frac{\lambda}{\alpha z_0}\int_{z_0}^{z_0 e^{\alpha t}} \left(\frac{z}{z_0}\right)^{-\frac{\lambda}{\alpha}-1} e^{ikz} dz,$$

from which we can read off the distribution function,

$$F_Z(z,t) = e^{-\lambda t} F_Z(ze^{-\alpha t},0) + (1-e^{-\lambda t})G(z,t),$$

where $G(z,t)$ is a truncated Pareto distribution,

$$G(z,t) = \begin{cases} 0, & z < z_0, \\ \frac{1}{1-e^{-\lambda t}}\left[1 - \left(\frac{z}{z_0}\right)^{-\frac{\lambda}{\alpha}}\right], & z_0 \le z \le z_0 e^{\alpha t}, \\ 1, & z > z_0 e^{\alpha t}. \end{cases}$$

As $t \to \infty$, the distribution $F_Z(z,t)$ approaches a Pareto distribution

$$F_Z(z,\infty) = 1 - \left(\frac{z}{z_0}\right)^{-\frac{\lambda}{\alpha}}, \quad z \ge z_0.$$

A closely related model of deterministic exponential growth with exponential stopping time was analyzed in [6]. We note by passing that the proportional growth is critical in generating a power law. Had the growth term in (8) been $\alpha Z_t^\delta \mathrm{d}t$ for some $\delta \in [0,1)$, the resulting distribution would have been a left truncated Weibull distribution with distribution function

$$F_Z(z,\infty) = 1 - \exp\left\{-\frac{\lambda}{\alpha(1-\delta)}(z^{1-\delta} - z_0^{1-\delta})\right\}, \quad z \ge z_0.$$

III. SDE driven by both Brownian motion and Poisson Counter

In this section, we add a Brownian motion component to (8), which becomes

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t + (x_0 - X_{t-})\mathrm{d}N_t, \tag{11}$$

where $\mu, x_0 \in \mathbb{R}$, $\sigma > 0$, W is a standard Brownian motion and N is a Poisson process with density λ, independent of W. This is a geometric Brownian motion with Poisson jumps which always reset the motion to a fixed state x_0. A similar model was analyzed in Reed [5].

Let $Y_t = \log X_t$ and $y_0 = \log x_0$. Then Itô's formula gives

$$\mathrm{d}Y_t = \left(\mu - \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma \mathrm{d}W_t + (y_0 - Y_{t-})\mathrm{d}N_t,$$

which is a Brownian motion randomly reset to y_0 by Poisson jumps. Let $\psi_k(y) = e^{iky}$ as in the previous section. By Itô's formula,

$$\begin{aligned}\mathrm{d}\psi_k(Y_t) = ik\psi_k(Y_t)&\left[\left(\mu - \frac{1}{2}\sigma^2\right)\mathrm{d}t + \sigma\mathrm{d}W_t\right] \\ &- \frac{1}{2}\sigma^2 k^2 \psi_k(Y_t)\mathrm{d}t + (e^{iky_0} - \psi_k(Y_{t-}))\mathrm{d}N_t.\end{aligned}$$

Taking expectations, we get

$$\begin{aligned}\frac{\partial}{\partial t}\Phi_Y(k,t) = &\left[i\left(\mu - \frac{1}{2}\sigma^2\right)k - \frac{1}{2}\sigma^2 k^2 - \lambda\right]\Phi_Y(k,t) \\ &+ \lambda e^{iky_0},\end{aligned}$$

where $\Phi_Y(k,t)$ is the characteristic function of Y_t. The solution is

$$\begin{aligned}\Phi_Y(k,t) = &\Phi_Y(k,\infty) \\ &+ e^{-\lambda t}[\Phi_Y(k,0) - \Phi_Y(k,\infty)]e^{i(\mu t - \frac{1}{2}\sigma^2 t)k - \frac{1}{2}\sigma^2 t k^2},\end{aligned}$$

where

$$\Phi_Y(k,\infty) = \frac{-\lambda e^{iky_0}}{i(\mu - \frac{1}{2}\sigma^2)k - \frac{1}{2}\sigma^2 k^2 - \lambda}.$$

Now we can find the steady-state density of Y_t as $t \to \infty$ by taking the inverse Fourier transform of $\Phi_Y(k,\infty)$,

$$f_Y(y) = \begin{cases} \frac{\alpha\beta}{\alpha+\beta} e^{\beta(y-y_0)}, & y \le y_0, \\ \frac{\alpha\beta}{\alpha+\beta} e^{-\alpha(y-y_0)}, & y \ge y_0, \end{cases}$$

where $\alpha > 0$ and $-\beta < 0$ are the two roots of the following quadratic equation,

$$\frac{1}{2}\sigma^2\gamma^2 + \left(\mu - \frac{1}{2}\sigma^2\right)\gamma - \lambda = 0. \tag{12}$$

Going back to X, we get the steady-state density of X_t as $t \to \infty$

$$\begin{aligned} f_X(x) &= f_Y(\log x) x^{-1} \\ &= \begin{cases} x_0^{-1}\frac{\alpha\beta}{\alpha+\beta}\left(\frac{x}{x_0}\right)^{\beta-1}, & x \in (0, x_0], \\ x_0^{-1}\frac{\alpha\beta}{\alpha+\beta}\left(\frac{x}{x_0}\right)^{-\alpha-1}, & x \in [x_0, \infty). \end{cases} \end{aligned} \tag{13}$$

which is the double Pareto distribution of Reed [5].

Motivated by the connection between (7) and (8), we also consider the following SDE

$$\mathrm{d}Z_t = \mu Z_t \mathrm{d}t + \sigma Z_t \mathrm{d}W_t - \frac{Z_{t-}^2}{Z_{t-} + \varepsilon}\mathrm{d}N_t, \tag{14}$$

with $Z_0 > 0$. Let $U_t = Z_t^{-1}$. Then, by Itô's formula,

$$\mathrm{d}U_t = -(\mu - \sigma^2)U_t \mathrm{d}t - \sigma U_t \mathrm{d}W_t + \varepsilon^{-1}\mathrm{d}N_t.$$

By the same procedure as before, we get the equation for the characteristic function $\Phi_U(k,t)$ of U,

$$\begin{aligned}\frac{\partial}{\partial t}\Phi_U(k,t) = &-(\mu - \sigma^2)k\frac{\partial}{\partial k}\Phi_U(k,t) \\ &+ \frac{1}{2}\sigma^2 k^2 \frac{\partial^2}{\partial k^2}\Phi_U(k,t) + (e^{ik/\varepsilon} - 1)\Phi_U(k,t),\end{aligned}$$

which is the Fourier transform with respect to the variable y of the following Fokker-Planck equation for the density $f_Y(y,t)$ of Y_t,

$$\begin{aligned}\frac{\partial}{\partial t}f_Y(y,t) = (\mu - \sigma^2)\frac{\partial}{\partial y}[y f_Y(y,t)] + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial y^2}[y^2 f_Y(y,t)] \\ + \lambda f_Y(y - \varepsilon^{-1}, t) - \lambda f_Y(y,t).\end{aligned}$$

In steady state, the density $f_Y(y)$ satisfies

$$\begin{aligned}(\mu - \sigma^2)\frac{\mathrm{d}}{\mathrm{d}y}[y f_Y(y)] + \frac{1}{2}\sigma^2\frac{\mathrm{d}^2}{\mathrm{d}y^2}[y^2 f_Y(y)] \\ + \lambda f_Y(y - \varepsilon^{-1}) - \lambda f_Y(y) = 0.\end{aligned}$$

For $y \in (0, \varepsilon^{-1}]$, this reduces to

$$(\mu - \sigma^2)\frac{\mathrm{d}}{\mathrm{d}y}[yf_Y(y)] + \frac{1}{2}\sigma^2\frac{\mathrm{d}^2}{\mathrm{d}y^2}[y^2 f_Y(y)] - \lambda f_Y(y) = 0. \quad (15)$$

The general solution to (15) is given by

$$f_Y(y) = Cy^{\alpha-1} + Dy^{-\beta-1}, \quad y \in (0, \varepsilon^{-1}],$$

where α and β are as in (13). The integrability of $f_Y(y)$ requires that $D = 0$, so

$$f_Y(y) = Cy^{\alpha-1}, \quad y \in (0, \varepsilon^{-1}].$$

Therefore,

$$f_X(x) = f_Y(x^{-1})x^{-2} = Cx^{-\alpha-1}, \quad x \in [\varepsilon, \infty),$$

which has the same upper tail power law exponent as in (13). This is intuitive since the difference of the two models lies in the range of small x. In fact, both (11) and (14) have the same general form

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \mathrm{d}W_t + [g(X_{t-}) - X_{t-}]\mathrm{d}N_t, \quad (16)$$

where $0 < g(\cdot) < B$ is bounded. The density of X_t satisfies the same Fokker-Planck equation for $x \in (B, \infty)$,

$$\frac{\partial}{\partial t}f_X = -\frac{\partial}{\partial x}(\mu x f_X) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(\sigma^2 x^2 f_X) - \lambda f_X.$$

Therefore, the upper tail of the steady state density has the same functional form, independent of $g(\cdot)$. By taking the limit $\sigma \to 0$, we observe that the remark here also applies to (7) and (8).

We note also the possibility of extending (16) to the multivariate case,

$$\mathrm{d}X_i = \mu_i X_i \mathrm{d}t + \sigma_i X_i \mathrm{d}W_i + \sum_j [g_{ij}(\mathbf{X}_-) - X_{i-}]\mathrm{d}N_j, \quad (17)$$

where we have omitted the subscript t. As in (16), each X_i has a power law upper tail. Thus (17) generates a set of correlated random variables with power law marginal distributions.

IV. SDE driven by bi-directional Poisson counters

In this section, we consider the following SDE driven by bi-directional Poisson counters,

$$\mathrm{d}X_t = \alpha(\mu - X_t)\mathrm{d}t + \sigma_1 \mathrm{d}M_t - \sigma_2 \mathrm{d}N_t,$$

where $\mu \in \mathbb{R}$, $\alpha, \sigma_1, \sigma_2 > 0$, and M, N are two independent Poisson processes with intensities λ_1, λ_2, respectively. By a simple shift of the origin, we may assume without loss of generality that $\mu = 0$ and the equation then becomes

$$\mathrm{d}X_t = -\alpha X_t \mathrm{d}t + \sigma_1 \mathrm{d}M_t - \sigma_2 \mathrm{d}N_t. \quad (18)$$

The characteristic function $\Phi_X(k,t)$ satisfies the following equation,

$$\begin{aligned}\frac{\partial}{\partial t}\Phi_X(k,t) = &- \alpha k \frac{\partial}{\partial k}\Phi_X(k,t) \\ &+ \left[\lambda_1(e^{ik\sigma_1} - 1) + \lambda_2(e^{-ik\sigma_2} - 1)\right]\Phi_X(k,t),\end{aligned} \quad (19)$$

the solution to which is

$$\begin{aligned}&\Phi_X(k,t) \\ =&\Phi_X(ke^{-\alpha t}, 0)e^{\frac{\lambda_1}{\alpha}\int_{\sigma_1 e^{-\alpha t}}^{\sigma_1}\frac{e^{iku}-1}{u}du - \frac{\lambda_2}{\alpha}\int_{-\sigma_2}^{-\sigma_2 e^{-\alpha t}}\frac{e^{iku}-1}{u}du}.\end{aligned}$$

Therefore,

$$\Phi_X(k,\infty) = e^{\frac{\lambda_1}{\alpha}\int_0^{\sigma_1}\frac{e^{iku}-1}{u}du - \frac{\lambda_2}{\alpha}\int_{-\sigma_2}^{0}\frac{e^{iku}-1}{u}du}.$$

This result can also be obtained using Theorem 17.5 of [7]. Lemma 2 of [8] shows that $\Phi_X(k,\infty)$ is the characteristic function belonging to an absolutely continuous distribution and the density is continuous if and only if $\lambda_1 + \lambda_2 > \alpha$. If $\lambda_1 = \lambda_2 = \lambda$ and $\sigma_1 = \sigma_2 = \sigma$, the steady-state distribution will be symmetric around the origin. If, in addition, $\sigma = \sigma_0 \lambda^{-\frac{1}{2}}$, then as $\lambda \to \infty$, (18) converges to the Ornstein-Uhlenbeck process and

$$\Phi_X(k,\infty) \to \exp\left\{-\frac{\sigma_0^2}{2\alpha}k^2\right\},$$

i.e. the characteristic function of the normal distribution $\mathcal{N}(0, \frac{\sigma_0^2}{\alpha})$, as expected.

Setting the right-hand side of (19) to zero, we get the differential equation satisfied by $\Phi_X(k,\infty)$,

$$\alpha k \frac{\mathrm{d}}{\mathrm{d}k}\Phi_X(k,t) = \left[\lambda_1(e^{ik\sigma_1} - 1) + \lambda_2(e^{-ik\sigma_2} - 1)\right]\Phi_X(k,t).$$

Thus the steady-state density satisfies the following equation,

$$\alpha\frac{\mathrm{d}}{\mathrm{d}x}[xf(x)] + \lambda_1 f(x-\sigma_1) - (\lambda_1+\lambda_2)f(x) + \lambda_2 f(x+\sigma_2) = 0. \quad (20)$$

Figure 1 shows some steady-state densities for (18) obtained from simulation, where we have set $\sigma_1 = \sigma_2 = 1$ and $\lambda_1 = \lambda_2$. Note that as $\frac{\lambda_1+\lambda_2}{\alpha}$ becomes larger, the density becomes smoother, consistent with Lemma 2 of [8]. As $\frac{\lambda_1+\lambda_2}{\alpha}$ becomes smaller, the density becomes more sharply concentrated around zero. In the case $\lambda_1 + \lambda_2 \ll \alpha$, $f(x-\sigma_1)$ and $f(x+\sigma_2)$ are negligible compared to $f(x)$ for small x and, hence, (20) can be approximated by the following equation,

$$\alpha\frac{\mathrm{d}}{\mathrm{d}x}[xf(x)] - (\lambda_1 + \lambda_2)f(x) = 0.$$

which can then be solved to give

$$f(x) = C|x|^{\frac{\lambda_1+\lambda_2}{\alpha}-1}, \quad 0 < |x| \ll 1.$$

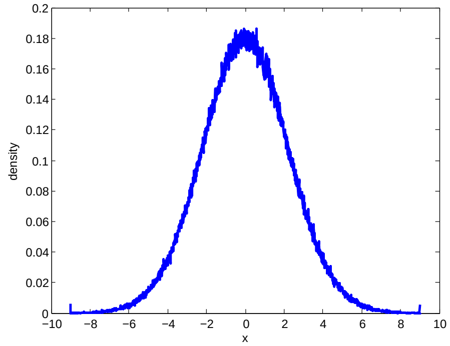
Figure 2 plots the steady-state densities in log-log scale with the reference lines of slope $\frac{\lambda_1+\lambda_2}{\alpha} - 1$ superimposed. The approximation is very good near the origin.

This approximation can be made more rigorous using Theorem 53.8 of [7]. We will present a simplified and more straightforward analysis of the behavior of the density near the origin when $\lambda_1 + \lambda_2 \le \alpha$. Let

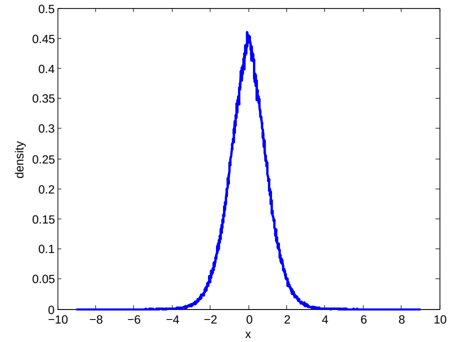
$$\Phi_1(k) = \exp\left\{\frac{\lambda_1}{\alpha}\int_0^{\sigma_1}\frac{e^{iku}-1}{u}du\right\}$$

and

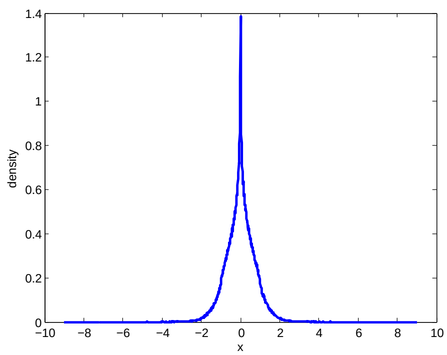
$$\Phi_2(k) = \exp\left\{-\frac{\lambda_2}{\alpha}\int_{-\sigma_2}^{0}\frac{e^{iku}-1}{u}du\right\},$$



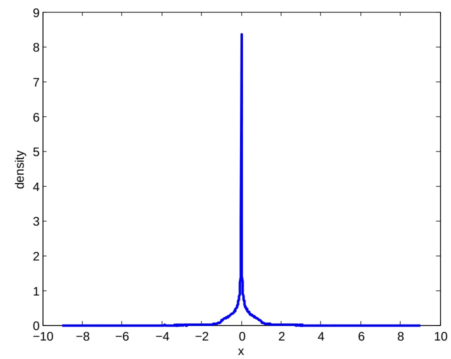
(a) $\frac{\lambda_1+\lambda_2}{\alpha} = 10$.



(b) $\frac{\lambda_1+\lambda_2}{\alpha} = 2$.

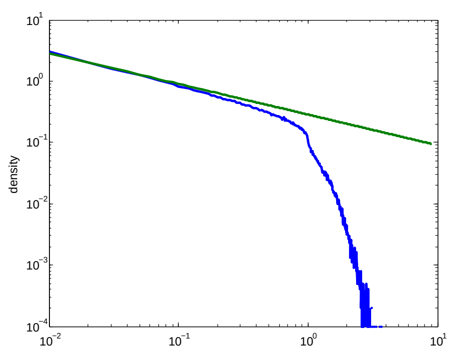


(c) $\frac{\lambda_1+\lambda_2}{\alpha} = 1$.

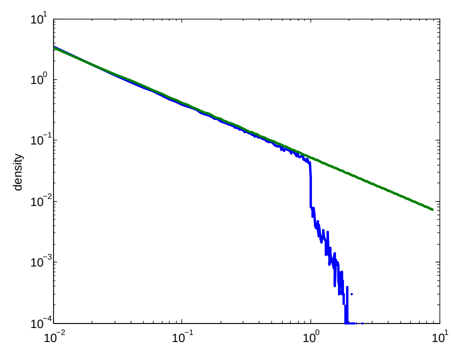


(d) $\frac{\lambda_1+\lambda_2}{\alpha} = 0.5$.

Fig. 1. Steady-state densities of X in (18); $\sigma_1 = \sigma_2 = 1$, $\lambda_1 = \lambda_2$.



(a) $\frac{\lambda_1+\lambda_2}{\alpha} = 0.5$.



(b) $\frac{\lambda_1+\lambda_2}{\alpha} = 0.1$.

Fig. 2. Log-log plot of steady-state densities of X in (18); $\sigma_1 = \sigma_2 = 1$, $\lambda_1 = \lambda_2$. The reference straight lines have a slop of $\frac{\lambda_1+\lambda_2}{\alpha} - 1$.

which are the characteristic functions of two absolutely continuous distributions. Denote their densities by $g(x)$ and $h(x)$, respectively. Then [8] shows that $g(x)$ has support on $[0, \infty)$ and $g(x) = Cx^{\frac{\lambda_1}{\alpha}-1}$ for $x \in (0, \sigma_1]$. Similarly, $h(x)$ has support on $(-\infty, 0]$ and $h(x) = D|x|^{\frac{\lambda_2}{\alpha}-1}$ for $x \in [-\sigma_2, 0)$. Since $\Phi_X(k, \infty) = \Phi_1(k)\Phi_2(k)$, the density $f(x)$ corresponding to $\Phi_X(k, \infty)$ is given by

$$f(x) = \int_{-\infty}^{\infty} g(y)h(x-y)\mathrm{d}y.$$

Let $m = \min\{\sigma_1, \sigma_2\}$ and $\epsilon \in (0, m)$. For $0 < x \le m - \epsilon$, we have

$$\begin{aligned} f(x) &= \int_x^{\infty} g(y)h(x-y)\mathrm{d}y \\ &= \int_x^m Cy^{\frac{\lambda_1}{\alpha}-1} D(y-x)^{\frac{\lambda_2}{\alpha}-1}\mathrm{d}y \\ &\quad + \int_m^{\infty} g(y)h(x-y)\mathrm{d}y \\ &= CDx^{\frac{\lambda_1+\lambda_2}{\alpha}-1}\int_1^{\frac{m}{x}} u^{\frac{\lambda_1}{\alpha}-1}(u-1)^{\frac{\lambda_2}{\alpha}-1}\mathrm{d}u \\ &\quad + \int_m^{\infty} g(y)h(x-y)\mathrm{d}y \\ &= A_1 x^{\frac{\lambda_1+\lambda_2}{\alpha}-1} + A_2, \end{aligned}$$

where $A_1 = CD\int_1^{\frac{m}{x}} u^{\frac{\lambda_1}{\alpha}-1}(u-1)^{\frac{\lambda_2}{\alpha}-1}\mathrm{d}u$ and $A_2 = \int_m^\infty g(y)h(x-y)\mathrm{d}y$. It is shown in [9] that $g(x)$ is continuous on $\mathbb{R}\setminus\{0\}$ and $g'(x)\le 0$, so it is uniformly bounded on $[m,\infty)$ and hence

$$A_2 \le \sup_{y\ge m} g(y)\int_m^\infty h(x-y)\mathrm{d}y \le \sup_{y\ge m} g(y) < \infty.$$

If $\lambda_1+\lambda_2<\alpha$, then

$$A_1 \ge CD\int_1^{\frac{m}{m-\epsilon}} u^{\frac{\lambda_1}{\alpha}-1}(u-1)^{\frac{\lambda_2}{\alpha}-1}\mathrm{d}u > 0$$

and

$$A_1 < CD\int_1^{\infty} u^{\frac{\lambda_1}{\alpha}-1}(u-1)^{\frac{\lambda_2}{\alpha}-1}\mathrm{d}u < \infty.$$

A similar analysis applies for the case $x\in(-m+\epsilon,0)$. Therefore,

$$f(x) = \Theta\left(|x|^{\frac{\lambda_1+\lambda_2}{\alpha}-1}\right), \quad \text{as } x\to 0.$$

If $\lambda_1+\lambda_2=\alpha$,

$$\begin{aligned} A_1 &= CD\int_1^{\frac{m}{x}} u^{\frac{\lambda_1}{\alpha}-1}(u-1)^{\frac{\lambda_2}{\alpha}-1}\mathrm{d}u \\ &\ge CD\int_1^{\frac{m}{x}} u^{\frac{\lambda_1}{\alpha}-1}u^{\frac{\lambda_2}{\alpha}-1}\mathrm{d}u = CD\log\frac{m}{x}, \end{aligned}$$

and for $x\le\frac{m}{2}$,

$$\begin{aligned} A_1 &= CD\int_1^{\frac{m}{x}} u^{\frac{\lambda_1}{\alpha}-1}(u-1)^{\frac{\lambda_2}{\alpha}-1}\mathrm{d}u \\ &\le CD\int_1^{2}(u-1)^{\frac{\lambda_2}{\alpha}-1}\mathrm{d}u + CD\int_2^{\frac{m}{x}} u^{\frac{\lambda_1}{\alpha}-1}\left(\frac{u}{2}\right)^{\frac{\lambda_2}{\alpha}-1}\mathrm{d}u \\ &= CD\frac{\alpha}{\lambda_2} + CD2^{1-\frac{\lambda_2}{\alpha}}\log\frac{m}{2x}. \end{aligned}$$

A similar analysis applies for $x<0$. Therefore,

$$f(x) = \Theta(-\log|x|), \quad \text{as } x\to 0.$$

V. Conclusions

We presented some simple stochastic differential equations that lead to lower tail and/or upper tail power law behaviors. Some of the results are known but the derivations are different. We also presented a model with two opposite Poisson counters and an exponential decaying term. This model exhibits power law behavior near a critical point, which might be of interest to statistical physics.

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